J.T. Lyczak, February 2016

Trace fields of knots

These are the knotes from the seminar on knot theory in Leiden in the spring of 2016. The website and all the knotes for this seminar can be found at `http://pub.math.leidenuniv.nl/~lyczakjt/seminar/knots2016.html`.

In these knotes we will first define hyperbolic knots. The article [Thu82] will be the main reference for this section. After that we will consider general manifolds with a hyperbolic structure and in particular determine the covering space for complete hyperbolic manifolds. The well-written notes [Lac00] were the main source for this section. Next, we will use the hyperbolic structure to find a representation of the knot group into $\mathrm{PSL}_2(\mathbb{C})$, which will allow us to define the trace field for hyperbolic knots. We will show that this invariant is actually a number field following the proof in [MR03].

1 Thurston's trichotomy

Let us begin with the definition promised two weeks ago.

Definition 1. A *hyperbolic knot* is a knot in S^3 whose complete has a complete hyperbolic structure.

This was the missing link in the anticipated classicification of knots.

Theorem 1.1 (Thurston's trichotomy)**.** *Every knot is either*

- *a torus knot, e.g. the trefoil;*
- *a satellite knot;*
- *a hyperbolic knot.*

The theorem is a consequence of the following theorem.

Theorem 1.2 (Thurston's hyperbolization theorem)**.** *Let $\bar{M}$ be a compact 3-manifold bounded by a torus, i.e. $\delta\bar{M} = S^1 \times S^1$. Assume that $\pi_1(\delta\bar{M})$ maps injectively into $\pi_1(\bar{M})$ and any subgroup $\mathbb{Z}^2 \subseteq \pi_1(\bar{M})$ is conjugate to $\pi_1(\delta\bar{M}) \subseteq \pi_1(\bar{M})$. Then the interior $M \subseteq \bar{M}$ has a complete hyperbolic structure of finite volume.*

So when we have a knot in S^3 we can thicken it to a torus as we have seen several times before. The M we would like to consider is the complement of this torus and $\bar{M}$ is the complement of such an open torus.

Fact 1.3. *If M satisfies the conditions of the lemma, one can always decompose M into ideal tetrahedra, as we did with the complement of the figure-8 knot last week.*

By Mostow rigidity we have that this hyperbolic structure if it exists is the unique one which is both complete and of finite volume.

Using the following terminology, one can more easily phrase the above theorem.

Definition 2. Let S be surface embedded in the complement $S^3\backslash K$. We say that S is *an essentially embedded surface* if the $\pi_1(S)$ maps injectively into $\pi(K)$.

So the condition in the hyperbolization theorem basically says that there are no essentially embedded surfaces of genus 1 apart from the one 'parallel' to the boundary. One can show that this condition is equivalent to saying that the knot is not a torus and not a satellite knot. This shows that the trichotomy indeed follows from the hyperbolization theorem.

2 The universal cover of a hyperbolic manifold

Let M be a connected manifold. We consider its universal cover $\pi : \widetilde{M} \to M$. One can think of $\widetilde{M}$ as the space of homotopy classes (with respect to the end points) of paths in M starting at a choosen base point p in M. The map π simply sends a homotopy class $[\alpha]$ to $\alpha(1)$. So the fiber above a $q \in M$ consists of homotopy classes of paths in M from p to q.

If M is a hyperbolc manifold, then $\widetilde{M}$ has the structure of a hyperbolic manifold making $\pi : \widetilde{M} \to M$ into a local isometry. Furthermore, if M is complete so is $\widetilde{M}$. Now any covering transformation, i.e. a homeomorphism $\widetilde{M} \to \widetilde{M}$ over M, is automatically a local isometry. The group of covering transformations is denoted by $\mathrm{Aut}(\pi)$.

We will first need the following result saying that local isometries of hyperbolic manifolds are uniquely determined on an open subset.

Lemma 2.1 ([Lac00], Lemma 10.2)**.** *Let M and N be hyperbolic manifolds and assume M is connected. Suppose we have two local isometries $f, g : M \to N$ such that f and g agree on a non-empty open subset in M. Then f and g agree on the whole of M, i.e. $f = g$.*

Proof. The proof uses the fact that two isometries of hyperbolic n-space are the same if they agree on a non-empty open subset. The trick is to define the subset of M consisting of the points $x \in M$ satisfying $f(x) = g(x)$. One can show this set is both open and closed in M, which concludes the proof since M is connected. Details can be found in [Lac00]. □

Now let α be a path in M and let (U_0, ϕ_0) be a fixed chart around $\alpha(0)$. We will produce a curve in $\mathbb{H}^n$ which is locally the image of α under some chart. To that end choose points

$$x_0 = \alpha(t_0), x_1 = \alpha(t_1), \ldots, x_r = \alpha(t_r)$$

such that $0 = t_0 < t_1 < \ldots < t_r = 1$, such that two subsequent points x_i, x_{i+1} lie in the domain of a chart $\phi_i : U_i \to \mathbb{H}^n$.

We have a unique isometry $g_{i,i+1}$ of $\mathbb{H}^n$ mapping $\phi_{i+1}(U_i \cap U_{i+1})$ to $\phi_i(U_i \cap U_{i+1})$. Now note that for t near t_1 the curves

$$t \mapsto \phi_0 \circ \alpha(t), \quad \text{and} \quad t \mapsto g_{01} \circ \phi_1 \circ \alpha(t)$$

coincide. So we can glue these part together to get a curve

$$t \mapsto \begin{cases} \phi_0 \circ \alpha(t) & \text{for } t_0 \le t \le t_1; \\ g_{01} \circ \phi_1 \circ \alpha(t) & \text{for } t_1 \le t \le t_2; \\ g_{12} g_{01} \circ \phi_2 \circ \alpha(t) & \text{for } t_2 \le t \le t_3; \\ \vdots & \vdots \\ g_{r-1,r} \ldots g_{21} g_{01} \circ \phi_r \circ \alpha(t) & \text{for } t_{r-1} \le t \le t_r. \end{cases}$$

Since the isometries $g_{i,i-1}$ are uniquely determined by a their restriction to a non-empty open subset, we get the following result.

Lemma 2.2 (Analytic continuation)**.** *The curve defined above for a chart (U_0, ϕ_0) and a curve α is independent of the choice of covering and is well-defined up to homotopy (with respect to the end points).*
This curve is called the analytic continuation of the chart along α.

Let us again fix a chart $\phi_0 : U_0 \to \mathbb{H}^n$ around a point $x_0 \in \widetilde{M}$. We can use the analytic continuation to extend the chart into a local isometry $\widetilde{M} \to \mathbb{H}^n$.

Theorem 2.3 (Developing morphism)**.** *For a fixed point $x_0 \in \widetilde{M}$ and a chart $x_0 \in U_0 \to \mathbb{H}^n$ we get a developing map*

$$D_{x_0,\phi_0} : \widetilde{M} \to \mathbb{H}^n$$

which restricts on U_0 to the chart and sends a homotopy class $[\alpha] \in \widetilde{M}$ to the endpoint of the continuation of ϕ_0 along α.
If we change the chart or the base point, the developing maps change by an isometry of $\mathbb{H}^n$.

We can use the fact that a different choice of chart gives a different developing map. Let $\tau : \widetilde{M} \to \widetilde{M}$ be a covering transformation and let (U_0, ϕ_0) be a chart containing a fixed point x_0 as before. Then we see that $(\tau^{-1}(U_0), \phi_0 \circ \tau)$ is a chart around $\tau^{-1}x_0$. These charts give use two different developing maps:

$$D_0 = D_{x_0,\phi_0} \quad \text{and} \quad D_\tau = D_{\tau^{-1}x_0,\phi_0\circ\tau}.$$

So there must exist an isometry $g_\tau \in \mathrm{Isom}(\mathbb{H}^n)$ such that

$$g_\tau \circ D_0 = D_\tau.$$

Now the two compositions in the diagram

$$\begin{tikzcd} \widetilde{M} \arrow[r, "\mathrm{Id}"] \arrow[d, "D_0"] & \widetilde{M} \arrow[d, "D_\tau"] \\ \mathbb{H}^n \arrow[r, "g_\tau"] & \mathbb{H}^n. \end{tikzcd}$$

agree on the non-empty open subset $\tau^{-1}U_0$ of $\widetilde{M}$ so the diagram commutes by Lemma 2.1.

Note that given the choice of x_0 and the chart, this element g_τ is in fact unique as it is uniquely determined on a non-empty open subset of $\mathbb{H}^n$. This shows that we for two covering transformations σ and τ that we must have the following equality $g_\sigma g_\tau = g_{\sigma\tau}$.

Theorem 2.4 (Holonomy map)**.** *The above construction gives us the following homomorphism of groups*

$$\rho : \mathrm{Aut}(\pi) \to \mathrm{Isom}(\mathbb{H}^n).$$

This maps depends on the choice of initial base point and chart, but two such choices give holonomy maps which are conjugated by an element in $\mathrm{Isom}(\mathbb{H}^n)$.

One can show that if M is oriented, then the image of ρ lies in $\mathrm{Isom}^+(\mathbb{H}^n)$.

Now if M is complete we can even more precisely described the universal cover.

Theorem 2.5. *Let M be a complete hyperbolic n-manifold. Then any developing map $D : \widetilde{M} \to \mathbb{H}^n$ is an isometry.*
This gives an isometry

$$M = \mathbb{H}^n / \rho\pi_1(M).$$

Here we used that M is isometric to the quotient of $\widetilde{M}$ under the action of $\mathrm{Aut}(\pi)$ and that the group $\mathrm{Aut}(\pi)$ of covering automorphisms is isomorphic to the fundamental group $\pi_1(M)$.

Corollary 2.6. *Suppose M is complete. The homomorphism*

$$\rho : \mathrm{Aut}(\pi) \to \mathrm{Isom}^+(\mathbb{H}^n)$$

is injective and we get a representation of the fundamental group

$$\rho : \pi_1(M) \to \mathrm{PSL}_2(\mathbb{C}).$$

The corollary comes from the fact that the only element in $\mathrm{Aut}(\pi)$ with a fixed point is the identity. Furthermore, we see that the image $\rho\pi_1(M)$ is discrete in $\mathrm{PSL}_2(\mathbb{C})$.

In particular, for hyperbolic knots K we find an injective map

$$\rho : \Gamma \to \mathrm{PSL}_2(\mathbb{C}),$$

where we wrote Γ for the knot group.

Actually, one can produce such a map for all knots, but the map will in general not be injective.

3 The trace field

Using the map ρ one can define an invariant for a hyperbolic knot.

Definition 3 (Trace field)**.** Let $K \subseteq S^3$ be a hyperbolic knot with a corresponding representation $\Gamma \to \mathrm{PSL}_2(\mathbb{C})$. We define the *trace field* of K as

$$\mathbb{Q}(\operatorname{tr}\Gamma) := \mathbb{Q}(\pm \operatorname{tr}\gamma_i : \gamma_i \in \Gamma).$$

So we have defined a field extension of $\mathbb{Q}$ by adjoining the traces of the matrices in $\rho(\Gamma)$. Obviously, the trace of an element in $\mathrm{PSL}_2(\mathbb{C})$ is not well-defined, but we can lift the matrix to $\mathrm{SL}_2(\mathbb{C})$. This uniquely defines the trace up to sign. Of course, the choice of sign does not matter in the definition of the trace field.
Note, that the trace field does not depend on ρ, since ρ is uniquely defined up to conjugacy and conjugate matrices have the same trace.

The most important fact about trace fields is the following theorem.

Theorem 3.1. *Let $\mathbb{Q}(\operatorname{tr}\Gamma)$ be the trace field of a hyperbolic knot. Then $\mathbb{Q}(\operatorname{tr}\Gamma)$ has finite degree over $\mathbb{Q}$ and hence is a number field.*

The theorem does in fact hold for many more subgroups Γ of $\mathrm{PSL}_2(\mathbb{C})$ such that $\mathbb{H}^3/\Gamma$ is a manifold and hence has hyperbolic structure. We will give such groups a name.

Definition 4. A discrete subgroup of $\mathrm{PSL}_2(\mathbb{C})$ is called a *Kleinian group*.
The covolume of Γ is defined as the (possibly infinite) hyperbolic volume of the hyperbolic orbifold $\mathbb{H}^3/\Gamma$.

As stated above, the trace field for any Kleinian group is a number field. The proof uses many intermediate results. One of which is the following generalization of Mostow rigidity to general Kleinian groups.

Theorem 3.2 (Mostow rigidity)**.** *Suppose $\phi : \Gamma_1 \to \Gamma_2$ is as isomorphism of Kleinian groups of finite covolume. Then there exists an isomorphism $g \in \mathrm{Isom}(\mathbb{H}^3)$ such that ϕ is conjugating by g.*

Some properties of knot groups are shared by all Kleinian groups. Such as the following two results.

Theorem 3.3 (Theorem 1.2.10 in [MR03])**.** *A Kleinian group of finite covolume is finitely generated.*

This result allows us to invoke the Scott core theorem.

Theorem 3.4 (Main theorem in [Sco73])**.** *Let M be a 3-manifold with finitely generated fundamental group. Then M has a compact submanifold of dimension 3, such that the inclusion map induces an isomorphism on fundamental groups $\pi_1(N) \to \pi_1(M)$.*

Since 3-manifolds are homotopy equivalent to finite CW-complexes we get the following corollary.

Corollary 3.5. *Kleinian groups of finite covolume are finitely presented.*

We will simplify the proof by assuming Γ has the following property satisfied by knot groups.

Theorem 3.6 (Theorem 3.30b in [BZH14])**.** *Let K be a knot in S^3. Then we have that the knot group $\pi_1(S^3 \backslash K)$ is torsion-free.*

So know we will give a proof of Theorem 3.1 for any torsion-free Kleinian group Γ of finite covolume.

Proof of Theorem 3.1. We will show that there is a conjugacy class of Γ in $\mathrm{SL}_2(\mathbb{C})$ such that the field generated by the coefficients of elements $\gamma_i \in \Gamma$ is a number field.

Let us first recall that an element $\gamma \in \mathrm{PSL}_2(\mathbb{C})$ does not only act on the hyperbolic 3-space, but also on the sphere S^2_∞. If we identify this with the Riemann sphere $\hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$, then the action is given by the Möbius transformation:

$$\pm \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \left\{ z \mapsto \frac{az+b}{cz+d} \right\}.$$

It is known that each such maps (except the identity) have exactly two fixed points if we count them with multiplicity. In our situation we want to use the following lemma. We state a weak version, which is enough for our situation. The results does in fact hold for a larger class of subgroups of $\mathrm{PSL}_2(\mathbb{C})$.

Lemma 3.7 (Theorem 1.2.2 in [MR03])**.** *A torsion-free Kleinian group of finte covolume has infinitely many elements with two distinct fixed points on $\hat{\mathbb{C}}$, such that any two such elements have no common fixed points.*

Now lift Γ to a subgroup of $\mathrm{SL}_2(\mathbb{C})$. By abuse of notation we will still denote this group by Γ. Since Γ is finitely generated and presented we can now fix generators $\gamma_1, \gamma_2, \ldots, \gamma_r$ and relations

$$R_1(\gamma_1, \gamma_2, \ldots, \gamma_r) = R_2(\gamma_1, \gamma_2, \ldots, \gamma_r) = \ldots = R_s(\gamma_1, \gamma_2, \ldots, \gamma_r) = I.$$

We may assume, after possibly adding generators and relations, that γ_1 and γ_2 both have two distinct fixed points on $\hat{\mathbb{C}}$ which do not overlap. By conjugation we can assume that γ_1 fixes 0 and $\infty \in \hat{\mathbb{C}}$ and 1 is one of the fixed points of γ_2.

Now we construct an affine variety as follows:
Consider the complex affine space of dimension $4r$ with coordinates x_i, y_i, z_i, w_i for $1 \leq i \leq r$. We will write A_i for the matrix

$$A_i = \begin{pmatrix} x_i & y_i \\ z_i & w_i \end{pmatrix}.$$

The conditions that the A_i lies in $\mathrm{SL}_2(\mathbb{C})$, A_1 has 0 and ∞ as fixed points, A_2 fixes 1, and the A_i satisfy the same relations as the γ_i, i.e.

$$R_j(A_1, A_2, \ldots, A_s) = I$$

defines an algebraic variety. Each point of this variety gives us a subgroup of $\mathrm{SL}_2(\mathbb{C})$. Furthermore, the variety is non-empty as it contains the (lift to $\mathrm{SL}_2(\mathbb{C})$ of the) image of $\Gamma \to \mathrm{PSL}_2(\mathbb{C})$. Let us write $V(\Gamma)$ for this algebraic variety. We have the following theorem by Weil and Garland.

Theorem 3.8 ([Wei60] and [Gar67])**.** *Let Γ be a finitely generated subgroup of $\mathrm{SL}_2(\mathbb{C})$. For a point ι sufficiently close to $\rho \in V(\Gamma)$ we have that the associated subgroup in $\mathrm{PSL}_2(\mathbb{C})$ has finite covolume and and the associated subgroup is isomorphic to Γ.*

We can use this to prove that in our case $V(\Gamma)$ is in fact of dimension 0. Assume it is of positive dimension. Then there are many subgroups satisfying the condition of the theorem. Such a subgroup $\Gamma' \subseteq \mathrm{SL}_2(\mathbb{C})$ is isomorphic to Γ and by Mostow rigidity this isomorphism is conjugation by an element of $g \in \mathrm{SL}_2(\mathbb{C})$. This gives

$$g^{-1} A_1 g = A_1' \quad \text{and} \quad g^{-1} A_2 g = A_2'$$

and g permutes the fix points of both A_1 and A_2. For such permutations there are 4 choices and for each such choice there is at most one g. However, there are many such subgroups.
So indeed $V(\Gamma)$ must be of dimension 0 and as it is defined to be irreducible it must be a point. We also know this point is defined over $\mathbb{Q}$ and this gives that the coordinates lie in a number field k. These coordinates are actually the coefficients of the generator γ_i and hence elements of Γ lie in $M_2(k)$ and hence all their traces lie in k:

$$\mathbb{Q}(\operatorname{tr}\Gamma) \subseteq k. \qquad \square$$

One can explicitely compute the trace field if the representation is given.

Example 1. The knot group of the figure-8 knot is given by

$$\langle x, y \mid xyx^{-1}y^{-1} = yxy^{-1}x^{-1}y \rangle$$

and the representation from the hyperbolic structure is for example given by

$$x \mapsto \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad y \mapsto \begin{pmatrix} 1 & 0 \\ \zeta_6 & 1 \end{pmatrix}.$$

This shows that $\mathbb{Q}(\operatorname{tr}\Gamma) \subseteq \mathbb{Q}(\zeta_6)$. Since

$$xy \mapsto \begin{pmatrix} 1+\zeta_6 & 1 \\ 0 & 1 \end{pmatrix}$$

we see that $\zeta_6 \in \mathbb{Q}(\operatorname{tr}\Gamma)$. So we conclude that the trace field is simply

$$\mathbb{Q}(\zeta_6) = \mathbb{Q}(\sqrt{-3})$$

since $\zeta_6 = \frac{1+\sqrt{-3}}{2}$.

We can also use the decomposition into finitely many ideal tetrahedra to compute the trace field. Recall that we could assume each tetrahedron had its vertices at 0, 1, ∞ and z. The complex number z was called the shape of the tetrahedron, and tetrahedra of shapes z, $\frac{1}{1-z}$ and $1 - \frac{1}{z}$ are isomorphic. So the shape of an tetrahedron is actually not well-defined, but the following result is.

Lemma 3.9 (Corollary 4.2.2 in [MR03])**.** *Let K be a hyperbolic knot such that the complement is decomposed into ideal tetrahedra of shape z_i. The field*

$$\mathbb{Q}(z_i)$$

equals the trace field of K.

This result is not true for general Kleinian groups. For knot groups, and in particular the figure-8, knot it does however hold.

Example 2. We saw last week that the complement of the figure-8 knot can be decomposed into two ideal tetrahedra both of shape ζ_6. So the trace field is indeed

$$\mathbb{Q}(\zeta_6).$$

Now one might wonder, which field actually occur as trace fields. This is still open, but one has some partial results.

Lemma 3.10 (Theorem 5.6.4 in [MR03])**.** *Consider positive square-free integers $d_1, d_2, \ldots, d_r$. There exists a hyperbolic 3-manifold of finite volume whose trace field equals*

$$\mathbb{Q}(\sqrt{-d_1}, \sqrt{-d_2}, \ldots, \sqrt{-d_r}).$$

The article [Neu11] contains the statement and the conjecture that basically any non-real number field occurs as a trace field.

4 Application of the trace field

Any number field k has a subring which is plays the part $\mathbb{Z}$ does in the field $\mathbb{Q}$.

Definition 5. The ring of integers $\mathcal{O}_k$ consists of the elements in k which are a root a monic polynomial with integral coefficients.

As the name suggest, $\mathcal{O}_k$ really is a ring.

Example 3. Let d be a square-free integer. The ring of integers of the number field $\mathbb{Q}(\sqrt{d})$ equals

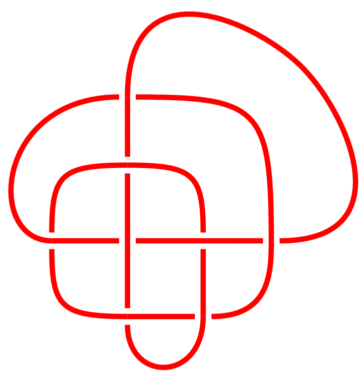
$$\mathcal{O}_k = \begin{cases} \mathbb{Z}[\sqrt{d}] & \text{if } d \equiv 1 \mod 4; \\ \mathbb{Z}[\frac{1+\sqrt{d}}{2}] & \text{if } d \not\equiv 1 \mod 4. \end{cases}$$

We can now use this notion to use the trace field to say something topological about the knot.

Theorem 4.1 (Bass's theorem)**.** *Suppose $M = \mathbb{H}^3/\Gamma$ is a hyperbolic manifold with finite volume. Suppose that Γ has an element whose trace does not lie in the ring of integers of the trace field. Then M contains a closed embedded essential surface.*

A proof can be found in section 5.2 of [MR03].

Example 4 (The knot 8_{16})**.** One can use Bass's theorem to prove that the following knot has an embedded essential surface. In fact it has genus 2. Can you find it?



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