Mastermath Elliptic Curves 2015

Remarks about the exam

The exam will test your knowledge and skills concerning

- the theory of all lectures except the final one (on the BSD conjecture), and
- *all* homework problems except
  - the ones marked as "optional" and
  - the ones that require a computer (i.e., Problem 63 and most, but not all, of the problems from the Sage worksheet).

The practice exam below is mostly a selection of (literal copies of) homework problems and is merely an indication of what the exam will be like. The actual exam will make a different selection of homework problems and replace them by variants (different questions, different wordings, different numbers, etc.).

The exam is *closed book*. You may use (but will probably not need) a non-programmable, non-graphing calculator and no other electronic equipment. You may not use any notes or books that you bring yourself. Extracts from books may be provided by us as appendices to the exam, for example when it comes to complicated formulas.

Prove all your claims, refer to theorems by name when possible (e.g., "the Nagell-Lutz theorem", "Hasse's bound") or in another way otherwise (e.g., "the homework exercise about ..." or "the list of properties of the dual homomorphism"), and explicitly verify *all* the hypotheses of a theorem when using it.

You have 3 hours for the exam, the exam date and location is announced by Mastermath.

Bring *identification* to the exam, it will be checked by VU University staff!

Practice exam

Problem 1. Let k be a field with $\mathrm{char}(k) \neq 7$ and let K be the Klein curve: $X^3Y + Y^3Z + Z^3X = 0$ in $\mathbf{P}^2$.

(a) Show that K is smooth.

(b) Compute the divisors of the functions X/Y and X/Z.

Problem 2. Let E be the elliptic curve over $\mathbf{F}_2$ given by $Y^2 + Y = X^3$.

(a) Compute the dual of its Frobenius endomorphism Frob.

(b) Compute the degree of Frob + [1].

Problem 3. Let k be a field. For convenience we assume that it is algebraically closed. Let C be a smooth projective curve defined over k and let $D = \sum_Q n_Q Q$ be a divisor of C.

(a) Show that $l(D) = 0$ if $\deg(D) < 0$.

(b) Let P be a point on C and let $n = \mathrm{ord}_P(D) := n_P$. Let $\pi \in k(C)$ be a uniformizer for P. In other words, we have $\mathrm{ord}_P(\pi) = 1$. Show that the map

$$L(D) \longrightarrow k$$

given by $f \mapsto (\pi^n f)(P)$ is k-linear and show that its kernel is $L(D - P)$.

(c) Show that for all divisors D and all points P we have

$$\big(l(D) - l(D - P)\big) \in \{0, 1\}.$$

On other words, adding a point to a divisor will add at most one to the dimension of its linear system.

(d) Deduce from (a) and (c) that for every divisor D we have

$$l(D) \leq \max\{0, \deg D + 1\}.$$

Problem 4. Let E be the elliptic curve

$$Y^2 = X(X^2 + 3X + 5)$$

over $\mathbf{Q}$. Describe the group structure, including generators, of $E(\mathbf{Q})^{\mathrm{tors}}$.

Problem 5. Compute the rank of the the elliptic curve

$$E : Y^2 = X(X - 1)(X + 2)$$

over $\mathbf{Q}$. You may use [Cassels] §14, which is provided as an appendix to this exam. Hint: Show $E(\mathbf{Q})/\widehat{\phi}(E'(\mathbf{Q})) \cong C_2$ and $E'(\mathbf{Q})/\phi(E(\mathbf{Q})) \cong C_2$ and compute the 2-torsion.

Appendix

As an appendix, you would find a copy of §14 of [Cassels] if this were an actual exam instead of a practice exam. You can download a copy yourself from the course web page and add it to this exam. `http://pub.math.leidenuniv.nl/~lyczakjt/teaching/EC2015-files/cassels-14.pdf`