The Loughran-Smeets Conjecture over an arbitrary base

by

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based on work with Browning and Sarapin

Browning group working seminar

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Plan

I Local solubility in families

II General base

III Proofs

I Local solubility in families

Classical question • k # field

• X/k variety

Is $X(k)$ empty?

Exa $2x_0^2 + 7x_1^2 - 3x_2^2 = 0$. and how does this depend on $2, 7, -3$?

Our question • X, Y k-varieties

• $\pi : X \to Y$

For how many $y \in Y(k)$ is $X_y(k)$ empty?

Exa $X = \{ y_0 x_0^2 + y_1 x_1^2 + y_2 x_2^2 = 0 \} \longrightarrow \mathbb{P}^2_k$

$\longmapsto (y_0 : y_1 : y_2)$

Dfn

$$N_{glob}(B) := \# \{ y \in Y(\mathbb{Q}) : X_y(k) \neq \emptyset, \ H_Y(y) \leq B \}$$

$$N_{loc}(B) := \# \{ y \in Y(\mathbb{Q}) : X_y(\mathbb{A}_k) \neq \emptyset, \ H(y) \leq B \}$$

Assumptions

- X, Y projective, geom. int
- Y smooth, X preferably too
- π dominant
- π has geom. int. generic fibre (a general X_y is geom. int.)

A bad member X_y does not influence the count by itself

But if there are enough of them they do.

Heuristic

The order of $N_{loc}(B)$ is influenced by bad behaviour in codimension 1 on Y.

Exa (Poonen & Voloch)

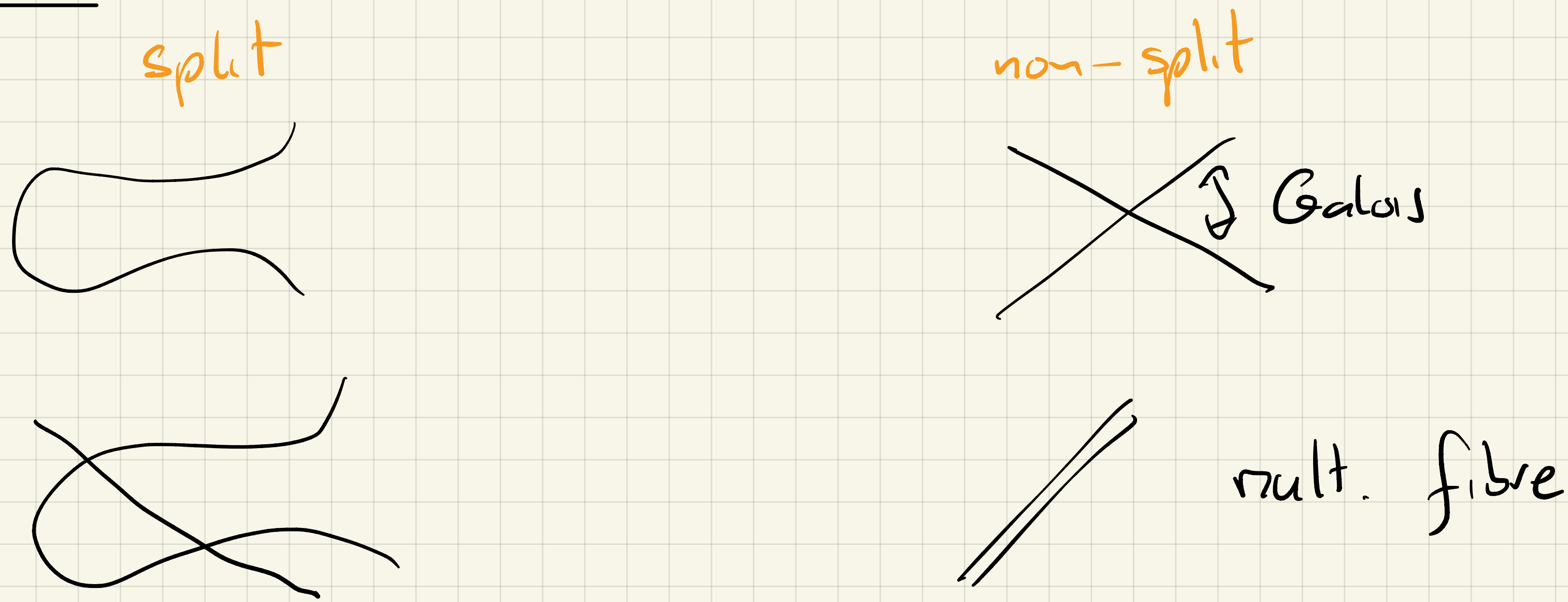
$$X = \{ y_0 x_0^d + y_1 x_0^{d-1} x_1 + \_\_\_\_\_\_ + y_N x_n^d \} \xrightarrow{\pi} \mathbb{P}^N, \quad N = \binom{n+d}{n} - 1$$

degree d hypersurfaces $X_y \subseteq \mathbb{P}^d$, $\quad n, d \geq 2 \quad (n,d) \neq (2,2)$.

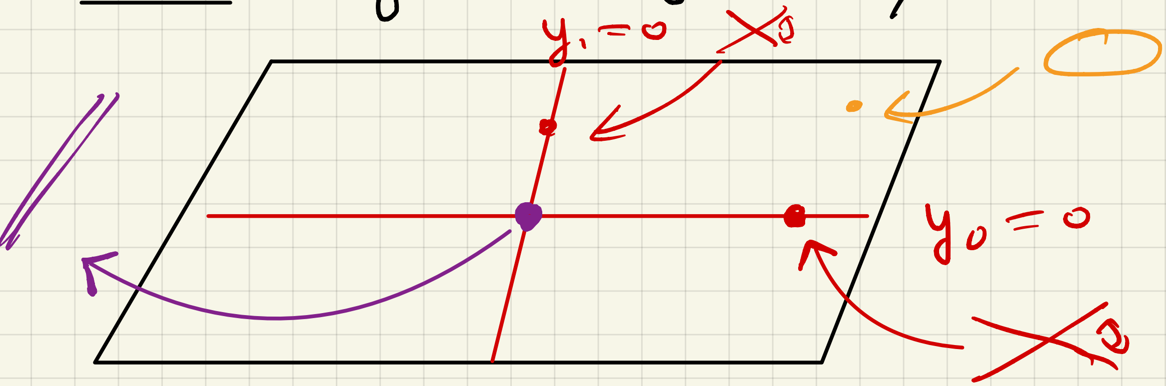
$N_{loc}(B) \sim c_\pi B \quad$ same order as $N_Y(B)$.

Dfn A variety X/k is split if it contains a non-empty geom. int. open $U \subseteq X$.

Exa



Exa $y_0 x_0^2 + y_1 x_1^2 + \overset{1}{\cancel{y_2}} x_2^2 = 0$ $\qquad y \in \mathbb{P}^2(k)$



For every codim 1 pt D we have a rat. number $0 \le \delta(D) \le 1$ which measures splitness.

<u>Properties</u>

- X_D split $\Rightarrow$ $\delta(D) = 1$
- X_D split by a cyclic field of degree $d \Rightarrow \delta(D) = \frac{1}{d}$

<u>Dfn</u> $$\Delta(\pi) = \sum_{D \in Y^{(1)}} (1 - \delta(D))$$

<u>Thm</u> (Loughran – Smeets) If $Y = \mathbb{P}^n$

$$N_{loc}(B) \ll \frac{B}{(\log B)^{\Delta(\pi)}}$$

<u>Conj</u> (Loughran – Smeets) If $Y = \mathbb{P}^n$ and no mult fibres over codim 1 pts, $B/(\log B)^{\Delta(\pi)}$ is the correct order of growth

II General base

What if $Y \neq \mathbb{P}^n$?

We still wat Y to have

- many rational pts
- no accummulating subsets

Naive conj Y Fano with ↑ then

$$N_{loc}(B) \sim c_\pi \frac{\boxed{B(\log B)^{\rho_Y - 1}}}{(\log B)^{\Delta(\pi)}}$$

← Manin conj for Y

Wrong!

$X = \{ y_0 x_0^2 + y_1 x_1^2 + y_2 x_2^2 + y_3 x_3^2 = 0 \}$ diag quad surfaces

$$\Big\downarrow$$

$Y = \{ y_0 y_1 = y_2 y_3 \}$ quad surfaces.

Non-split fibres over:

$D_{ij} = \{ y_i = y_j = 0 \}$ $i \in \{0,1\}$, $j \in \{2,3\}$.

<u>Note</u> • X is <u>not</u> smooth

- Let $\check{X} \to X$ be a desingularisation.
- We actually care about $\check{\pi} : \check{X} \to X \xrightarrow{\pi} Y$
- $N_{loc}(\pi, B)$ and $N_{loc}(\check{\pi}, B)$ have same order.
- $\Delta(\check{\pi})$ might be different from $\Delta(\pi)$, but is independent of desing. $\check{X}$.

<u>Thm</u> (Browning, Sarapin, L.)

$$B \ll N_{loc}(B) \ll B$$

But $\Delta(\tilde{\pi}) = 2$.

<u>Note</u> $Y \cong \mathbb{P}^1 \times \mathbb{P}^1$ so $\rho_Y = 2$, so naive conj. is wrong!

However the thin set $\{\vec{y} : -y_j/y_i \text{ is a square}\}$

already contributes to $N_{loc}(B)$ of order B.

<u>Remark</u> $y_0 x_0^2 + y_1 x_1^2 + y_2 x_2^2 = 0$ over $y_0 y_1 = y_2 y_3$

has same behaviour, but we lose the symmetry in y_2 & y_3.

Note the apparant double fibres over D_{02} and D_{12}

These disappear on $\check{X} \to Y$.

III Proofs

Prop The blowup $X' \to X$ resolves the singular locus above the generic points of D_{ij}.

Prf Reduce the blowup of the 4-dim X in the 1-dim locus into a blowup of a 3-dim variety in a 0-dim locus and compute explicitly.

Cor The fibre of X' over generic pts. of D_{ij} has 4 comp. which are conjugated in pairs

$$\Rightarrow \delta_{\tilde{\pi}}(D_{ij}) = \frac{1}{2} \Rightarrow \Delta(\tilde{\pi}) = 2.$$

Lem $N_{loc}(B) \gg B$.

Prf Consider $\bar{y} = (x : y : -x : -y)$. This already gives order B; count points $\mathbb{P}^1$ with the anticanonical height.

Prop $N_{loc}(B) \ll B$

Take $H(\bar{y}) = \max\{|y_i|\}^{2}$ ← anticanonical height.

1) $\#\{\bar{y} : y_0 y_1 y_2 y_3 = 0,\ H(\bar{y}) \leq B\} \sim cB$

2) $\bar{y} \in \mathbb{Z}^4_{\neq 0}$ can be written as

$$\bar{y} = (t_0 t_2,\ t_1 t_3,\ t_0 t_3,\ t_1 t_2), \qquad \bar{t} \in \mathbb{Z}^4_{\neq 0}$$

$$N_{loc}(B) \ll \#\left\{ \bar{t} \in \mathbb{Z}^4_{\neq 0} : \begin{array}{l} |t_0 t_2|, \ \_\_ \ \leq \sqrt{B} \\ X_{t_0 t_2, \_}(\mathbb{A}_{\mathbb{Q}}) \neq \emptyset \end{array} \right\} + O(B)$$

3) $\mathcal{U} = \{ \bar{u} \in \mathbb{Z}^4 : U_i/2 < u_i \le U_i \}$

$$N^*(U_0, \ldots, U_3) := \#\left\{ \bar{u} \in \mathcal{U} : \begin{array}{l} u_i \text{ square free} \\ X_{u_0 u_1, \ldots}(\mathbb{A}_\mathbb{Q}) \neq \emptyset \end{array} \right\}$$

Lem $N^*(\underline{U}) \ll \dfrac{U_0 U_1 U_2 U_3}{(1 + \log U_{min})^2}$ $\qquad U_{min} = \min\{U_i\}$.

Lem (Multidimensional large sieve, Kowalski)

$X = \{ \bar{n} \in \mathbb{Z}^d : -N_i \le n_i \le N_i \}$ and $\Omega_p \subseteq \mathbb{F}_p^d$

$$\#\{ \bar{n} \in X : \bar{n} \text{ does not reduce to } \Omega_p \text{ for all } p \le L \} \ll \frac{\prod_i (N_i + L^2)}{F(L)}$$

where $$F(L) = \sum_{n \le L} \mu^2(n) \prod_{p \mid n} \frac{|\Omega_p|}{p^d - |\Omega_p|}$$

Take Ω_p union of the set and its four syndrical brothers.

$$\{\bar{r} \in \mathbb{F}_p^4 : r_0 = 0,\ r_1 r_2 r_3 \neq 0,\ -r_2 r_3 \notin \mathbb{F}_p^{\times,2}\}.$$

<u>Lem</u> $\bar{u}$ squarefree and $\bar{u}$ reduces to $\Omega_p \Rightarrow X_{\bar{u}}(\mathbb{Q}_p) = \emptyset$.

<u>Prf</u> standard.

<u>Lem</u> $F(L) \gg (\log L)^2$

<u>Prf</u> $|\Omega_p|$ computable, so $\dfrac{|\Omega_p|}{p^4 - |\Omega_p|} \geq \dfrac{2}{p}\left(1 - \dfrac{2}{p}\right)^3$

Using (among others) $\sum_{p \leq X} \frac{2}{p}\left(1 - \frac{2}{p}\right)^3 \log p \sim 2 \log X$

$\Delta\left(\frac{n}{\pi}\right)$

we get $F(L) \gg (\log L)^2$.

<u>Prf for $N^*(\underline{u}) \ll$</u> $u_{min} < 2$ clear ✓

for $u_{min} \geq 2$ use large sieve with $L = \sqrt{u_{min}}$

$$N^*(\underline{u}) \ll \frac{u_0 u_1 u_2 u_3}{(\log u_{min})^2} \ll \frac{u_0 u_1 u_2 u_3}{(1 + \log u_{min})^2}$$

4) All $\underline{t}$ in an interval

$$N(\underline{T}) := \#\{\underline{t} \in \mathbb{Z}^4 : T_i/2 < t_i \leq T_i \, , \; X_{\underline{t}}(\mathbb{A}_{\mathbb{Q}}) \neq \phi\}$$

$t_i = u_i m_i^2 \qquad \mu(u_i)^2 = 1 . \Rightarrow \quad \boxed{X_{\underline{t}}(\mathbb{A}_{\mathbb{Q}}) = \phi \Leftrightarrow X_{\underline{u}}(\mathbb{A}_{\mathbb{Q}}) = \phi}$

$$N(\underline{T}) = \sum_{m_0 \leq \sqrt{T_0}} \text{---} \sum_{m_3 \leq T_3} N^*\left(\frac{T_0}{m_0^2}, \text{---}, \frac{T_3}{m_3^2}\right)$$

$$\ll \text{---} \text{''} \text{---} \sum_{0 \leq j \leq 3} \frac{T_0 T_1 T_2 T_3}{m_0^2 m_1^2 m_2^2 m_3^2 \left(1 + \lg(T_j/m_j)\right)^2}$$

$$\ll T_0 T_1 T_2 T_3 \sum_{0 \leq j \leq 3} \sum_{m_j \leq \sqrt{T_j}} \frac{1}{m_j^2 \left(1 + \lg(T_j/m_j)\right)^2}$$

For m large the sum is very small, the other region gives

$$\ll \frac{T_0 T_1 T_2 T_3}{1 + \log T_{min}}$$

5) Sum over T_i powers of 2 and $\ll \sqrt{B}$ gives $N_{loc}(B) \ll B$.