Nodal curves on surfaces

An application of algebraic cobordism

Julian Lyczak

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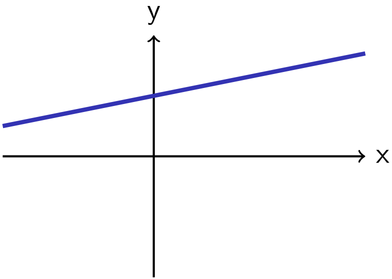
Curves

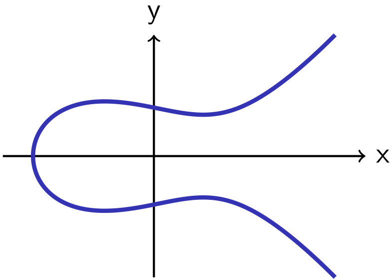
What is a curve?

What is a curve?



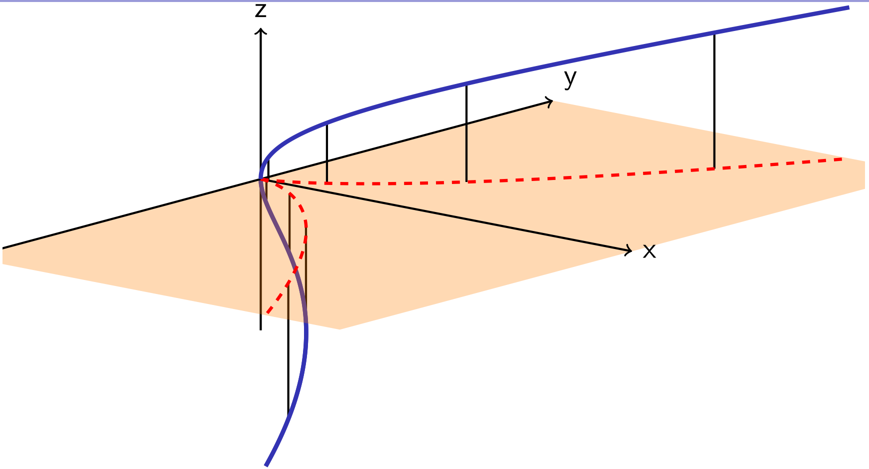
$$X + 5Z - 5Y$$

$$x + 5 = 5y$$



$$ZY^2 - X^3 + 2XZ^2 - 4Z^3$$

$$y^2 = x^3 - 2x + 4$$



$$Z^3 - XW^2,\ Z^2 - XW \text{ and } Y^2 - XZ$$

$$z^3 - y \text{ and } z^2 - x$$

Smooth curves

Definition of smoothness

A curve is **smooth** if it has a unique tangent line at every point.

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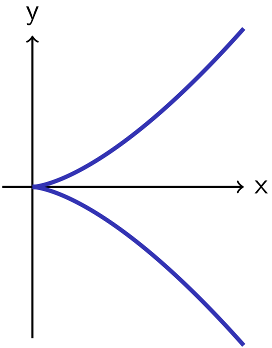
Condition for plane curves to be smooth

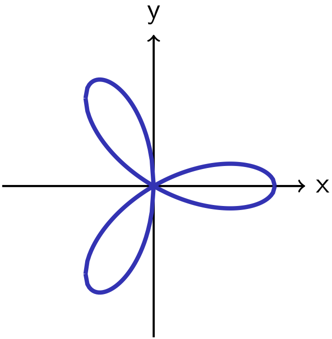
A plane curve given by $F(X, Y, Z) = 0$ is smooth if and only if the derivatives

$$\frac{dF}{dX}, \ \frac{dF}{dY} \ \text{and} \ \frac{dF}{dZ}$$

do not vanish simultaneously.

Singularities





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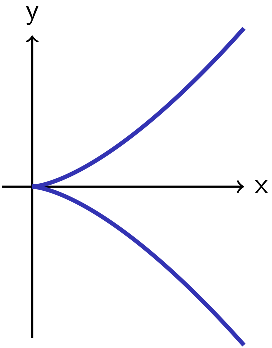
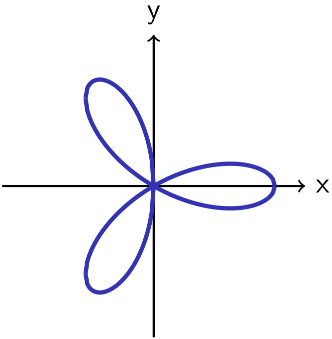


No tangents!



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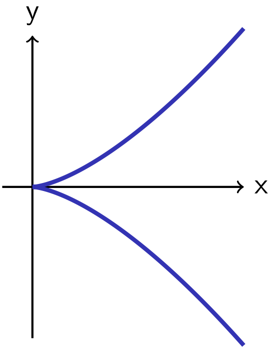

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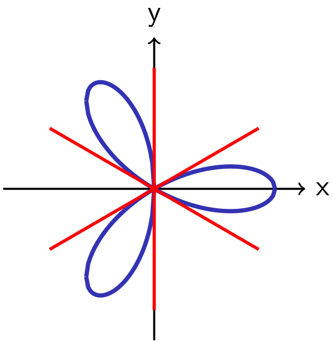
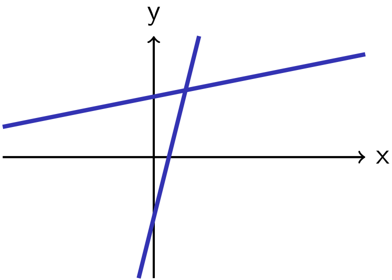
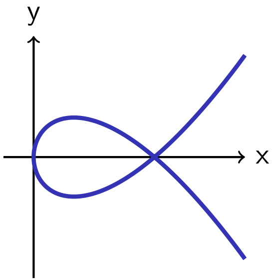


3 tangents!

Nodal curves





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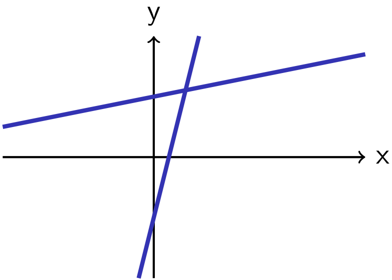
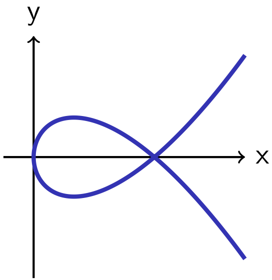


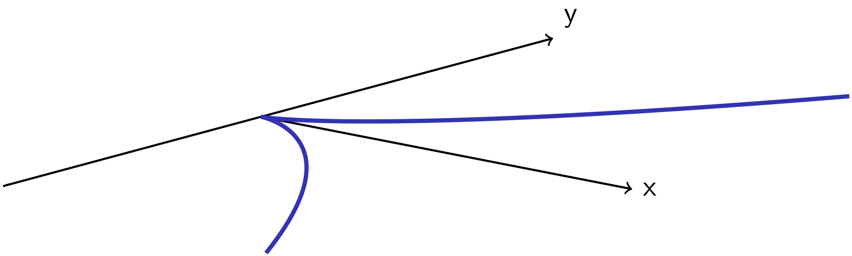


Nodal curves

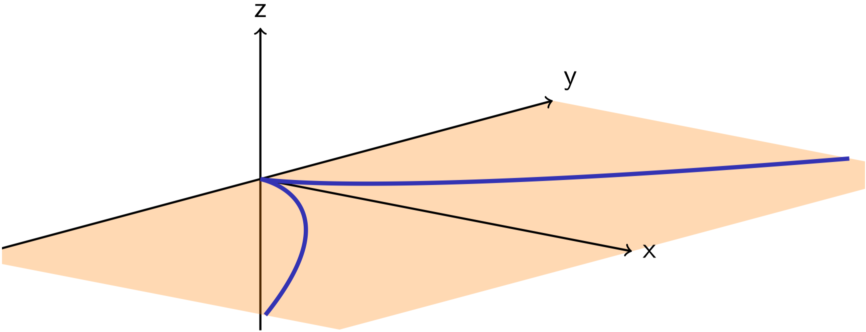
A point on a plane curve is called a **node** if it has two distinct tangent lines.

A **nodal curve** is a plane curve with only nodes as singularities.

Resolving the singularity



Resolving the singularity



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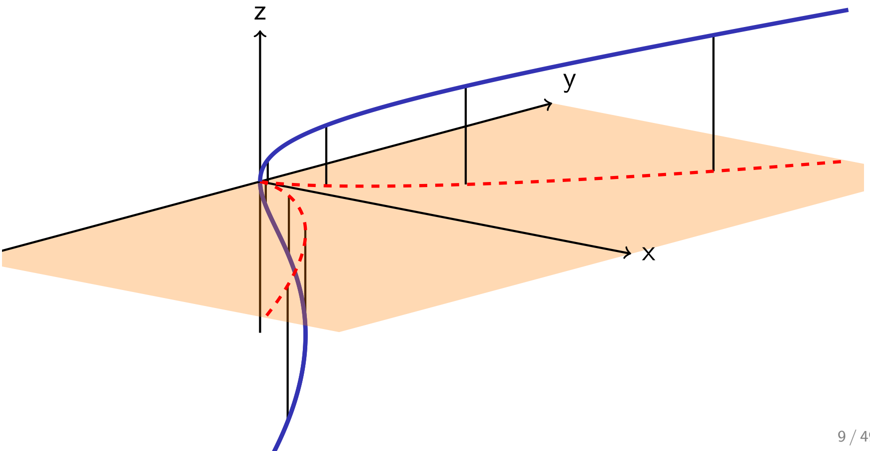

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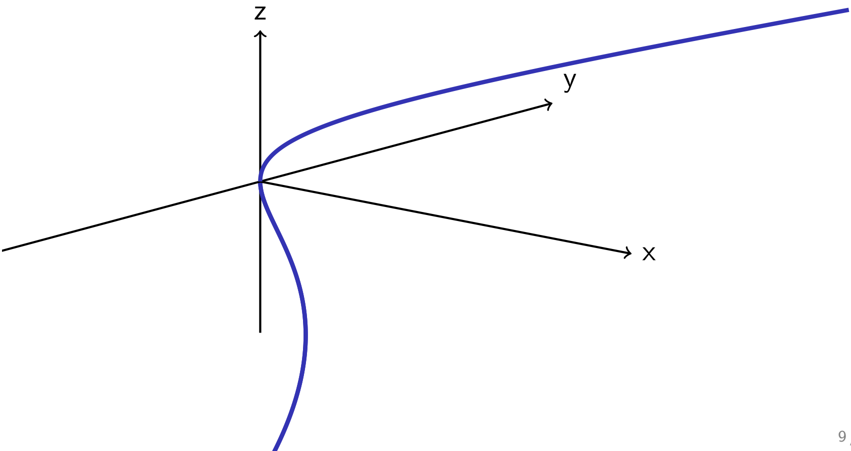

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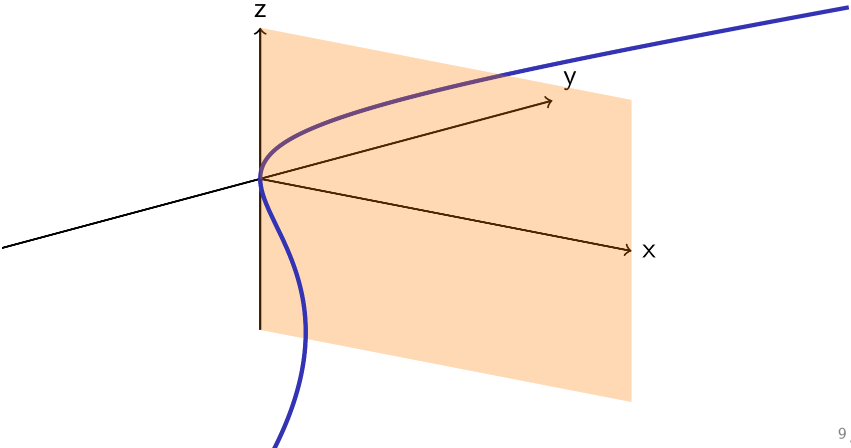

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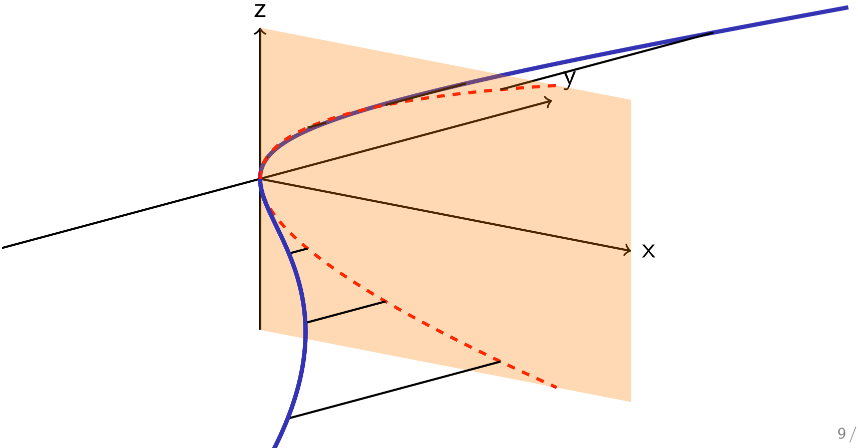

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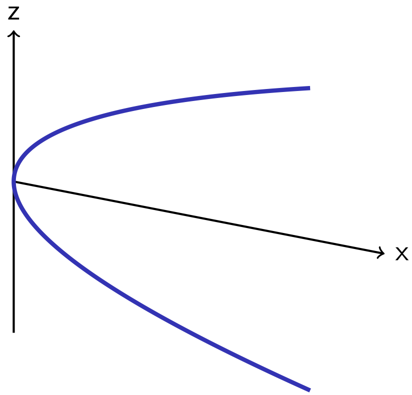

General curves

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Every curve looks like a nodal curve

Any curve is birational to a nodal curve in $\mathbb{P}^2$.

Nodal curves on $\mathbb{P}^2$:

Severi varieties

Space of all curves

Each tuple $(a_{i,j,k})_{i,j,k}$ with i, j and k non-negative with sum equal to d gives a plane curve

$$F(X,Y,Z) = \sum_{i+j+k=d} a_{i,j,k} X^i Y^j Z^k.$$

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Severi variety

The Zariski closure of all points in $\mathbb{P}^{\frac{d(d+3)}{2}}$ corresponding to nodal curves of degree d with exactly δ nodes is called the **Severi variety** of type d, δ.

Notation: $\mathcal{V}^{d,\delta}$.

Degree 2 curves

The space of all degree 2 curves is $\mathbb{P}^5$, as a curve is a linear combination of

$$AX^2 + BXY + CXZ + DY^2 + EYZ + FZ^2.$$

The only degree two curves are smooth conics and line pairs, so a degree 2 curve has at most one node.

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The only degree two curves are smooth conics and line pairs, so a degree 2 curve has at most one node.
It is smooth if and only if

$$\det \begin{pmatrix} 2A & B & C \\ B & 2D & E \\ C & E & 2F \end{pmatrix} \neq 0.$$

So $\mathcal{V}^{2,1}$ is of dimension 4.

Degree 3 curves

The space of all degree 3 curves is $\mathbb{P}^9$.

- 3 nodes: a union of three lines. Dimension: $2+2+2=6$;
- 2 nodes: a union of a line and a conic. Dimension: $2+5=7$;
- 1 node: an irreducible nodal curve. Dimension: ?
- 0 node: smooth cubic curve.

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- 0 node: smooth cubic curve.

Dimension of the Severi varieties

Severi (1921)

The Severi variety of type d, δ is non-empty for $0 \leq \delta \leq \frac{d(d-1)}{2}$ and has codimension δ.

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Harris (1958)

The irreducible nodal curves lie in a single irreducible component of the Severi variety.

Nodal curves on $\mathbb{P}^2$:

Severi degree

Degree of a plane curve

Consider the plane $\mathbb{P}^2$.

- Most lines intersect a degree d curve in d points.
- We could take this as the definition of the degree of a curve.

Degree of a closed subspace of a projective variety

A codimension *d* subspace intersects a general *d*-dimensional linear projective subspace in a finite number of point. This is the **degree** of the subspace.

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- Degree of a set of points is the number of these points.
- Degree of the total projective space equals 1.

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Severi degree

The degree of the Severi variety is called the **Severi degree**.

A plane in the space of all curves

How can we pick a general linear plane of a given dimension in the total space of all curves of degree d?

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Geometrical interpretation of Severi degree

The Severi degree $\mathcal{N}^{d,\delta}$ equals the number of degree d curves through $\frac{d(d+3)}{2} - \delta$ points.

Severi degree for $d = 2$

Consider the conics.

- Clearly $\delta = 0$ gives 1, as for any degree.
- Now for $\delta = 1$: how many line pairs pass through 4 points?

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- For $\delta = 3$: How many line triples pass through 6 points?
  $\mathcal{N}^{3,3} = 5 \cdot 3 \cdot 1 = 15$.

- For $\delta = 2$: How many combinations of a line and a conic pass through 7 points?
  $\mathcal{N}^{3,2} = \binom{7}{2} = 21$.

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So in our notation $\mathcal{N}^{3,1} = 12$.

Nodal curves on $\mathbb{P}^2$:

Severi degree as polynomials in d

Severi degree for $\delta = 2$ and $\delta = 3$

Cayley (1863)

Number of nodal curves of degree d through $\frac{d(d+3)}{2} - 2$ points is

$$\frac{3}{2}(d-1)(d-2)(3d^2-3d-11).$$

Severi degree for $\delta = 2$ and $\delta = 3$

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$$\frac{3}{2}(d-1)(d-2)(3d^2-3d-11).$$

Roberts (1876)

Number of nodal curves of degree d through $\frac{d(d+3)}{2} - 3$ points is

$$\frac{9}{2}d^6 - 27d^5 + \frac{9}{2}d^4 + \frac{423}{2}d^3 - 229d^2 - \frac{829}{2}d + 525$$

for $d \geq 3$.

Nodal polynomials

Di Fransesco's and Itzykson's conjecture (1994)

There exists polynomials $N_\delta(d)$ in d of degree 2δ, such that

$$N_\delta(d) = \mathcal{N}^{d,\delta}$$

for large enough d.

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It was proven for $\delta = 4$, 5 and 6 by Vainsencher in 1995.
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Fomin and Mikhalkin (2009)

The statement is true for all δ.

Nodal curves on a surface:

Vector bundles

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Note that on the space $\mathbb{P}^2$ we have a vector space $\mathcal{O}(d)$, whose elements give a value of $\mathbb{C}$ at each point of $\mathbb{P}^2$.

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Define a **vector bundle** of **rank** r on a surface S, as a vector space whose elements, called **global sections**, give us values in $\mathbb{C}^r$ at each point of S.

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Line bundles

A vector bundle $\mathcal{L}$ of rank 1 is called a **line bundle**.

The points on a surface S where a global section of $\mathcal{L}$ gives zero is a curve.
We call this a **curve in $\mathcal{L}$**.

k-ample line bundles

Assumption on line bundles

A line bundle $\mathcal{L}$ is called ***k*-ample** if for all $k+1$ points p_i and numbers $\alpha_i \in \mathbb{C}$, there exists a global section of $\mathcal{L}$ which gives α_i at p_i.

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The line bundle $\mathcal{O}(d)$ on $\mathbb{P}^2$ is $\frac{d(d+3)}{2} - 1$-ample.

Nodal curves on a surface:

Chern numbers

Definition of Chern subspaces

Consider a vector bundle $\mathcal{F}$ of rank r on a space X, then any $r - p + 1$ linearly independent general global sections s_i determine a subspace of codimension p, by

$$c_p(\mathcal{F}) := \{x \in S \mid \dim \mathrm{Span}\,(s_1(x), s_2(x), \ldots, s_{r-p+1}(x)) \leq r - p\}$$

depending on the sections. This is called a pth **Chern subspace** of $\mathcal{F}$.

The space $c_p(\mathcal{F})$ does not actually have to have codimension p, it can also be empty. As is the case for $p > r$.

Chern numbers

Intersection of Chern subspaces

For any two vector bundles $\mathcal{F}$ and $\mathcal{G}$ and integers p and q there are choices of sections such that

$$c_p(\mathcal{F})c_q(\mathcal{G}) := c_p(\mathcal{F}) \cap c_q(\mathcal{G})$$

is of codimension $p + q$.

Chern numbers

If we consider such a subspace

$$c_{p_1}(\mathcal{F}) \dots c_{p_s}(\mathcal{F}) c_{q_1}(\mathcal{G}) \dots c_{q_t}(\mathcal{G})$$

of dimension 0, it is simply a finite number of points. This is called a **Chern number** of $\mathcal{F}$ and $\mathcal{G}$ and is independent of all the chosen global sections!

Chern numbers on the projective plane

Consider $\mathcal{O}(d)$. It has only one Chern number: we have that $c_0(\mathcal{O}(d)) = X$ and $c_1(\mathcal{O}(d))$ is any curve of degree d. If we take another curve, for example just d lines, to compute

$$c_1(\mathcal{O}(d))^2 = c_1(\mathcal{O}(d))c_1(\mathcal{O}(d)) = d^2.$$

The only Chern numbers of $\mathcal{O}(d)$ and $\mathcal{O}(e)$ are d^2, de and e^2.

Nodal curves on a surface:

The number of δ-nodal curves

The generalized Severi degree

A section $s \in \mathcal{L}$ and a multiple λs for $\lambda \in \mathbb{C}$ give the same curve.

So curves in $\mathcal{L}$ correspond to points in a projective space $\mathbb{P}^{v-1} = \mathbb{P}(\mathcal{L})$, where v is the dimension of the vector space $\mathcal{L}$.

The number of δ-nodal curves

The number of curves with δ nodes coming from a global section in a general δ-dimensional plane in $\mathbb{P}(\mathcal{L})$ is called the **number of δ-nodal curves** in $\mathcal{L}$.

Göttsche showed that this is well-defined if $\mathcal{L}$ is $(5\delta - 1)$-ample.

Polynomials in Chern numbers

The tangent sheaf $\mathcal{T}_S$ of a surface is a two-dimensional vector bundle.

Theorem

The number of δ-nodal curves in a k-ample line bundle $\mathcal{L}$, for large enough k and δ, is a polynomial of degree δ in the Chern numbers of $\mathcal{L}$ and $\mathcal{T}_S$. These are

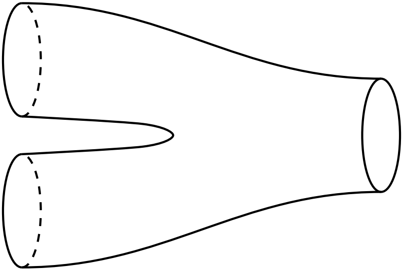
$$c_1(\mathcal{L})^2, \quad c_1(\mathcal{L})c_1(\mathcal{T}_S), \quad c_1(\mathcal{T}_S)^2 \quad \text{and} \quad c_2(\mathcal{T}_S).$$

The proof:

Algebraic cobordism

Classical cobordism theory

A **cobordism** is a manifold with a left and a right boundary. We say that two spaces are **cobordant** if they occur at the respective left and right boundary.



Algebraically cobordant

There are certain pairs of surfaces (W, Z) which are called **algebraically cobordant**.

Algebraic cobordism of surfaces, ω_2, is defined as the $\mathbb{Q}$-vector space of formal finite sums

$$\sum_i q_i [S_i],$$

for surfaces S_i, subject to the following two conditions:

- $[W] = [Z]$ for all algebraically cobordant W and Z;
- $[S_1 \coprod S_2] = [S_1] + [S_2]$ for all surfaces S_1 and S_2.

Algebraic cobordism with line bundles

There are certain pairs of surfaces with line bundles $(W, \mathcal{L}), (Z, \mathcal{M})$ which are called algebraically cobordant.

Algebraic cobordism of surfaces with line bundles, $\omega_{2,1}$, is defined as the $\mathbb{Q}$-vector space of formal finite sums

$$\sum_i q_i[S_i, \mathcal{L}_i],$$

for surfaces S_i with a line bundle $\mathcal{L}_i$, subject to the following two conditions:

- $[W, \mathcal{L}] = [Z, \mathcal{M}]$ for algebraically cobordant $(W, \mathcal{L})$ and $(Z, \mathcal{M})$;
- $[S_1 \coprod S_2, \mathcal{L}] = [S_1, \mathcal{L}|_{S_1}] + [S_2, \mathcal{L}|_{S_1}]$ for all surfaces S_1 and S_2 and a line bundle $\mathcal{L}$ on the disjoint union $S_1 \coprod S_2$.

Algebraic cobordism classes and Chern numbers

We can calculate Chern numbers for such sums of surfaces, or even surfaces with a line bundle, by calculating the analogous Chern number of all terms.

Algebraic cobordism is determined by Chern numbers

Any class $\sum q_i[S_i]$ in algebraic cobordism is uniquely determined by the Chern numbers of the respective tangent bundles.

So the class of $[S]$ is uniquely determined by $c_1(\mathcal{T}_S)^2$ and $c_2(\mathcal{T}_S)$.

Algebraic cobordism with line bundles by Chern numbers

Any class $\sum q_i[S_i, \mathcal{L}_i]$ in $\omega_{2,1}$ is uniquely determined by the Chern numbers of the tangent bundle and line bundles.

So the class of $[S, \mathcal{L}]$ is uniquely determined by $c_1(\mathcal{L})^2$, $c_1(\mathcal{L})c_1(\mathcal{T}_S)$, $c_1(\mathcal{T}_S)^2$ and $c_2(\mathcal{T}_S)$.

The proof:

Nodal polynomials

Generating function

For a surface S with a line bundle $\mathcal{L}$, Göttsche defined a space $S_{2\delta}$ of dimension 2δ and a vector bundle $\mathcal{L}_{3\delta}$ of rank 3δ.

Let $d_\delta(S, \mathcal{L})$ denote the number of points in the Chern number $c_{2\delta}(\mathcal{L}_{3\delta})$ and define $d_0(S, \mathcal{L}) = 1$.

Göttsche (1997)

If $\mathcal{L}$ is a $(5\delta - 1)$-ample line bundle on a surface, then $d_\delta(\mathcal{L})$ equals the number δ-nodal curves in $\mathcal{L}$.

Generating function of Chern numbers

We define

$$\phi(S, \mathcal{L}) = \sum_{k \geq 0} d_k(S, \mathcal{L}) x^k.$$

Tzeng (2012)

Let $S = S_1 \coprod S_2$ be the disjoint union of two surfaces with a line bundle $\mathcal{L}$ on S. Then we have

$$\phi(S, \mathcal{L}) = \phi(S_1, \mathcal{L}|_{S_1})\phi(S_2, \mathcal{L}|_{S_2}).$$

For any algebraically cobordant pair $(W, \mathcal{M}), (Z, \mathcal{N})$ of surfaces we have

$$\phi(W, \mathcal{M}) = \phi(Z, \mathcal{N}).$$

Group homomorphism

The two results of Tzeng give us exactly the following theorem.

Tzeng (2012)

The map ϕ extends to a linear map

$$\omega_{2,1} \to \mathbb{Q}[[t]]^*$$

of vector spaces over $\mathbb{Q}$.

Basis polynomials

By the map ϕ we get four polynomials $A(x)$, $B(x)$, $C(x)$ and $D(x)$ corresponding to the four Chern numbers, which form a basis for $\mathbb{Q}^4 \cong \omega_{2,1}$. So

$$\phi(S,\mathcal{L}) = A(x)^{c_1(\mathcal{L})^2} B(x)^{c_1(\mathcal{L})c_1(\mathcal{T}_S)} C(x)^{c_1(\mathcal{T}_S)^2} D(x)^{c_2(\mathcal{T}_S)}.$$

Writing $A(x) = 1 + a_1 x + a_2 x^2 + \dots$ and similarly for $B(x)$, $C(x)$ and $D(x)$, we get

$$\begin{aligned}
\phi(S, \mathcal{L}) &= (1 + a_1 x + \dots)^{\kappa} (1 + b_1 x + \dots)^{\lambda} \cdot \\
&\qquad (1 + c_1 x + \dots)^{\mu} (1 + d_1 x + \dots)^{\nu} \\
&= 1 + (a_1 \kappa + b_1 \lambda + c_1 \mu + d_1 \nu)\, x \\
&+ \left(a_2 \kappa + \binom{\kappa}{2} a_1^2 + b_2 \lambda + \binom{\lambda}{2} b_1^2 + \right. \\
&\qquad \left. c_2 \mu + \binom{\mu}{2} c_1^2 + d_2 \nu + \binom{\nu}{2} d_1^2 \right) x^2 \\
&+ \dots
\end{aligned}$$

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$d_1(S,\mathcal{L})$

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$d_1(S,\mathcal{L})$ $d_2(S,\mathcal{L})$

Nodal polynomials

If we calculate the coefficients, we get

$$d_1 = 3\kappa - 2\lambda + \nu,$$
$$d_2 = \frac{1}{2}\left(d_1(d_1 - 7) - 6\mu + 25\lambda - 21\kappa\right).$$

Nodal polynomials

The number d_k is a polynomial of degree k in κ, λ, μ and ν.

If $\mathcal{L}$ is a l-ample line bundle on S, then by Göttsche's result we see that for $0 \leq \delta \leq \lfloor \frac{l+1}{5} \rfloor$ we have that d_δ equals the number of nodal curves in a general δ-dimensional plane in $\mathbb{P}(\mathcal{L})$.

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Kool, Shende and Thomas even proved that the statement holds for all $0 \leq \delta \leq l$.

The Severi degree $\mathcal{N}^{d,1}$

Let us calculate Chern subspace for $\mathcal{T}_{\mathbb{P}^2}$.
Use $x\frac{d}{dx}$ and $y\frac{d}{dy}$ for $c_1(\mathcal{T}_{\mathbb{P}^2})$:
This gives the two axes and the line at infinity.

Use $x\frac{d}{dx} + y\frac{d}{dy}$ for $c_2(\mathcal{T}_{\mathbb{P}^2})$:
This vanishes precisely at $[1:0:0]$, $[0:1:0]$ and $[0:0:1]$.

So $c_1(\mathcal{L})^2 = d^2$, $c_1(\mathcal{L})c_1(\mathcal{T}_S) = 3d$, $c_1(\mathcal{T}_S)^2 = 9$ and $c_2(\mathcal{T}_S) = 3$.

This gives

$$d_1(\mathbb{P}^2, \mathcal{O}(d)) = 3\kappa - 2\lambda + \nu = 3d^2 - 2\cdot 3d + 3 = 3(d-1)^2.$$

Thank you for your attention

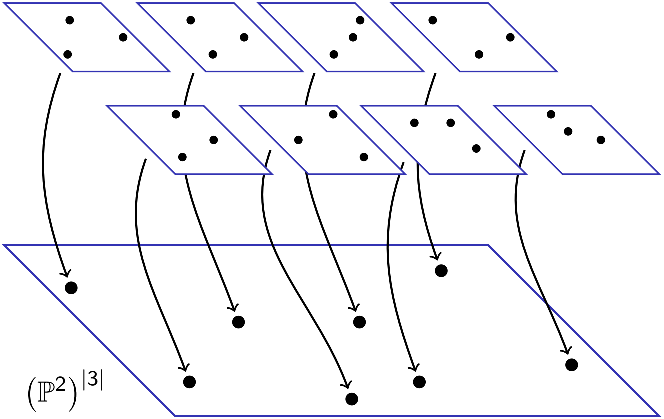
Table of Severi degrees

	1	2	3	4	5	6	7
$\mathcal{N}^{d,0}$	1	1	1	1	1	1	1
$\mathcal{N}^{d,1}$	0	3	12	27	48	75	108
$\mathcal{N}^{d,2}$	0	0	21	225	882	2370	5175
$\mathcal{N}^{d,3}$	0	0	15	675	7915	41310	145383
$\mathcal{N}^{d,4}$	0	0	0	666	36975	437517	2667375
$\mathcal{N}^{d,5}$	0	0	0	378	90027	2931831	33720354
$\mathcal{N}^{d,6}$	0	0	0	105	109781	12597900	302280963
$\mathcal{N}^{d,7}$	0	0	0	0	65949	34602705	1950179922
$\mathcal{N}^{d,8}$	0	0	0	0	26136	59809860	9108238023
$\mathcal{N}^{d,9}$	0	0	0	0	6930	63338881	30777542450
$\mathcal{N}^{d,10}$	0	0	0	0	945	40047888	74808824084

The table is from *Plane curves, node polynomials and floor diagrams*, F.S. Block's dissertation at the University of Michigan 2001.

Definition of Hilbert schemes

Let k be a non-negative integer and S a surface. A space $S^{|k|}$ whose points represent k points in S is called a **Hilbert scheme of points**.



Properties of Hilbert schemes

Any smooth projective surface has a unique Hilbert scheme of k points, and it is

- smooth;
- of dimension $2k$;
- proper.

Non-reduced points

What happens if two points coincide?

Non-reduced points

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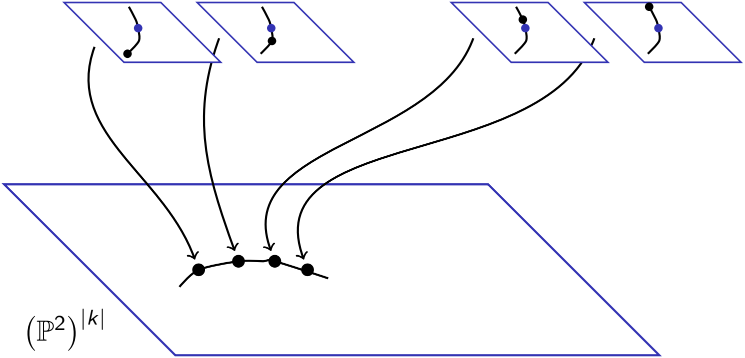

Non-reduced points

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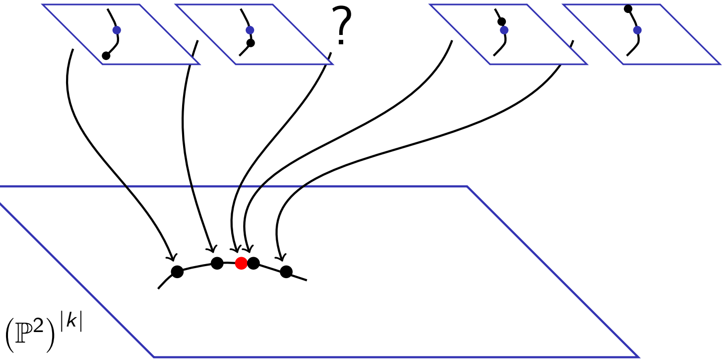

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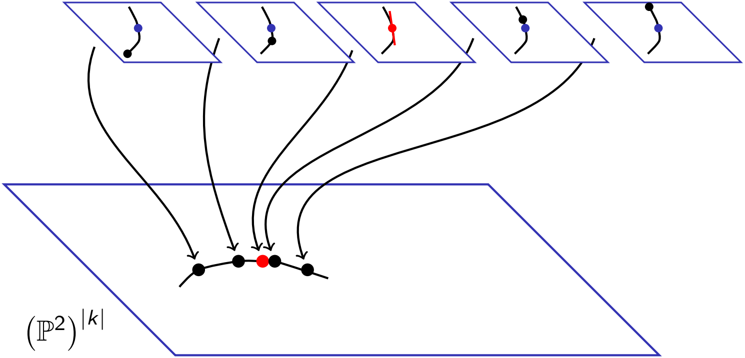

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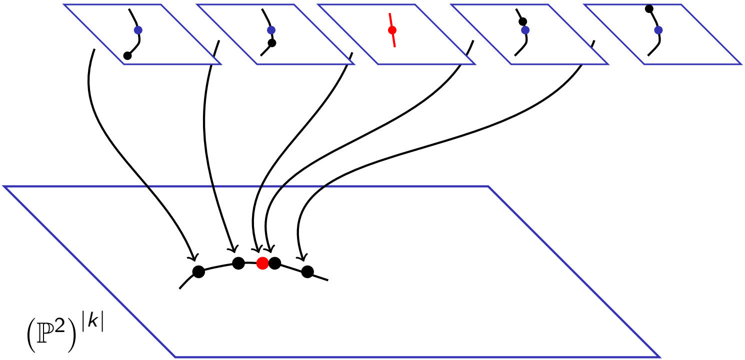

Degree d functions on Hilbert scheme of the projective plane

Consider the projective plane and the Hilbert scheme of k points.

- A degree d function can be evaluated at the k points represented by the point on the Hilbert scheme.
- This gives a vector bundle $\mathcal{O}(d)_k$ on the Hilbert scheme with the same global sections but of rank k!
- It will vanish if the corresponding degree d curve passes through these points.
- If it vanishes at a point representing a double point, the corresponding curve admits the corresponding tangent.

Similarly, a line bundle $\mathcal{L}$ on a surface S will give a vector bundle $\mathcal{L}_k$ on $S^{[k]}$ of rank k.

Singular points

To ensure we get singularities, we want a point with two distinct tangent direction with length zero!

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"Singular points" on the Hilbert scheme

Let us write $S_{2\delta}$ for the points in $S^{|3\delta|}$ consisting of δ points each given with two zero directions. It has dimension 2δ.

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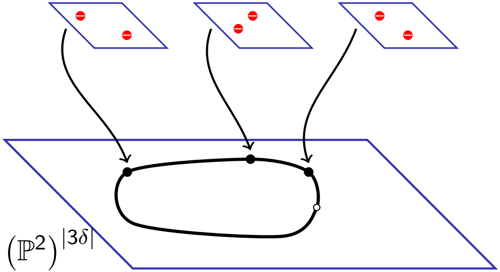

The number of δ-nodal curves

The vector bundle $\mathcal{L}_{3\delta}$ on the Hilbert scheme of 3δ points of a surface S restricts to a vector bundle on $S_{2\delta}$, the subspace of "singular points" on the Hilbert scheme.

Let $d_\delta(\mathcal{L})$ denote the number of points in the Chern number $c_{2\delta}(\mathcal{L}_{3\delta})$.

Göttsche (1997)

If $\mathcal{L}$ is a $(5\delta - 1)$-ample line bundle on a surface, then $d_\delta(\mathcal{L})$ equals the number δ-nodal curves in $\mathcal{L}$.