The comparison theorem

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1 The comparison theorem

1.1 Continuous maps of sites; the Leray spectral sequence

Let C be a site. We have the following properties:

- The inclusion functor $\mathbf{Ab}(C) \to \mathbf{PAb}(C)$ has a left adjoint, called *sheafification*;
- The category $\mathbf{Ab}(C)$ has enough injectives (so we can do sheaf cohomology!).

A *continuous map of sites* $f : C \to D$ is a functor $f^{-1} : D \to C$ such that for every covering $\{V_i \to V\}$ in D:

- the collection $\{f^{-1}V_i \to f^{-1}V\}$ is a covering in C;
- for any morphism $T \to V$ the natural morphisms $f^{-1}(V_i \times_V T) \to f^{-1}(V_i) \times_{f^{-1}(V)} f^{-1}(T)$ are isomorphisms.

The notations and directions of the arrows might seem confusing here, but make sense considering the following example.

Example 1.1.1. Let $f : X \to Y$ be a continuous map of topological spaces. Let X_{open} and Y_{open} be the sites of open subsets of X and Y, respectively. Then f induces a functor $f^{-1} : Y_{\text{open}} \to X_{\text{open}}$ that respects coverings and fibered products, and hence a continuous map of sites $f : X_{\text{open}} \to Y_{\text{open}}$.

Given a continuous map of sites $f : C \to D$ (or any functor $D \to C$ actually) and presheaf in $\mathbf{PAb}(C)$, we can define the *push-forward* f_pF in $\mathbf{PAb}(D)$ to be the presheaf given by $f_pF(U) = F(f^{-1}(U))$; it defined a functor $f_p : \mathbf{PAb}(C) \to \mathbf{PAb}(D)$. It has a left adjoint (the construction is similar to the construction of the pullback of a sheaf in the topological case) which we will denote by $f^p : \mathbf{PAb}(D) \to \mathbf{PAb}(C)$. So far we haven't used the continuity of f at all!

If F is an abelian sheaf on C, then it turns out that f_pF is a sheaf too. To avoid confusion we denote the induced functor $\mathbf{Sh}(C) \to \mathbf{Sh}(D)$ by f_*. This functor, too, has a left adjoint $f^* : \mathbf{Sh}(D) \to \mathbf{Sh}(C)$.

A *morphism of sites* $f : C \to D$ is a continuous map of sites such that the functor $f^* : \mathbf{Sh}(D) \to \mathbf{Sh}(C)$ is exact. The following proposition will allow us in many cases to check whether a continuous map of sites is a morphism.

Proposition 1.1.2 (Stacks Tag 00X6)**.** *Let $f : C \to D$ be a continuous map of sites. Assume the following:*

- *D has a terminal object X, and $f^{-1}(X)$ is a terminal object of C;*
- *D has fiber products, and f^{-1} commutes with them.*

Then f is a morphism of sites. □

Let $*$ denote the category consisting of 1 object and the identity morphism. We can endow $*$ with a Grothendieck topology in a unique way. We have $\mathbf{Ab}(*) = \mathbf{PAb}(*)$, and the global sections functor gives an equivalence of categories $\mathbf{PAb}(*) \to \mathbf{Ab}$.

Let C be a site and let U be an object of C. Then we have a functor $* \to C$ mapping the unique object of $*$ to U. This gives a continuous map of functors $f : C \to *$, and the push-forward $f_* : \mathbf{Ab}(C) \to \mathbf{Ab}(*)$ is left-exact. By composing this push-forward with the equivalence $\mathbf{Ab}(*) \to \mathbf{Ab}$ we find that the global sections functor $\Gamma_U : \mathbf{Ab}(C) \to \mathbf{Ab} : F \mapsto F(U)$ is left-exact, and we can therefore do sheaf cohomology! We define the functors $H^i(U, -) : \mathbf{Ab}(C) \to \mathbf{Ab}$ to be the right derived functors of Γ_U. If C has a terminal object X we define $H^i(C, -) = H^i(X, -)$.

Theorem 1.1.3 (Leray spectral sequence)**.** *Let $f : C \to D$ be a continuous map of sites. Then for every abelian sheaf F on C and every object V of D there exists a cohomological spectral sequence*

$$E_2^{pq} := H^p(V, R^q f_* F) \implies H^{p+q}(f^{-1}V, F).$$

Exercise 1.1.4. Prove this. Hint: use the Grothendieck spectral sequence.

Corollary 1.1.5. *Let $f : C \to D$ be a continuous map of sites. Let F be an abelian sheaf, and let V be an object of D. Suppose that $R^q f_* F = 0$ for all $q > 0$. Then*

$$H^p(V, f_* F) \cong H^p(f^{-1}V, F).$$

1.2 Complex analytic spaces

An *analytic subspace* of $\mathbb{C}^n$ is a locally ringed space $(Y, \mathcal{H}_Y)$ of the following form: let $U \subset \mathbb{C}^n$ be an (Euclidean) open subset, and let $f_1, \ldots, f_r$ be holomorphic functions on U. We let $Y \subset U$ denote the set of common zeroes of $f_1, \ldots, f_r$, and define $\mathcal{H}_Y = \mathcal{H}_U/(f_1, \ldots, f_r)$, where $\mathcal{H}_U$ is the sheaf of holomorphic functions on U.

A *complex analytic space* is a locally ringed space $(X, \mathcal{H}_X)$ which can be covered by open subsets, each of which is isomorphic as a locally ringed space to an analytic subspace of some $\mathbb{C}^n$. Often we omit $\mathcal{H}_X$ from our notation and simply write X.

A *morphism* or *holomorphic map* $X \to Y$ of complex analytic spaces is a morphism of locally ringed spaces.

An *analytic sheaf* on a complex analytic space X is a sheaf of $\mathcal{H}_X$-modules.

Notice that for every complex analytic space X there exists a natural morphism of locally ringed spaces $X \to \operatorname{Spec} \mathbb{C}$.

1.3 Covering spaces

Let X be a complex analytic space. Then X comes equipped with a topology, so we can define the site X_{cx} as the site of open subspaces of X.

We can define another site X_{cov} as follows. The objects of X_{cov} are complex analytic spaces Y together with a morphism $Y \to X$ which is a *local isomorphism*, that is, every point in Y has an open neighbourhood that is mapped isomorphically to an open subspace of X by the morphism $Y \to X$. The morphisms of X_{cov} are the morphisms of complex analytic spaces compatible with the fixed maps to X. A collection of morphisms $\{Y_i \to Y\}$ is a covering if and only if it is jointly surjective.

Exercise 1.3.1. Show that X_{cov} is, indeed, a site.

Any open subspace of X is a local isomorphism, so we get an inclusion functor $X_{\mathrm{cx}} \to X_{\mathrm{cov}}$.

Exercise 1.3.2. Prove that this functor defines a continuous map $X_{\mathrm{cov}} \to X_{\mathrm{cx}}$.

Proposition 1.3.3. *Let f be the continuous map $X_{\mathrm{cov}} \to X_{\mathrm{cx}}$. Then f_* is exact.*

Corollary 1.3.4. *Let F be a sheaf on X_{cov}. Then we have isomorphisms*

$$H^i(X_{\mathrm{cov}}, F) \cong H^i(X_{\mathrm{cx}}, f_* F).$$

So the study of cohomology of sheaves on X_{cov} is covered by that of cohomology of sheaves on X_{cx}. On the other hand, we will show that we can assign to every $\mathbb{C}$-scheme locally of finite type a complex analytic space, that under this 'analytification' functor étale morphisms correspond to local isomorphisms, and the sheaf cohomologies of $X_{\text{ét}}$ and X_{cov} are closely related.

1.4 Analytification

Let X be a scheme locally of finite type over $\mathbb{C}$. We will associate to X a complex analytic space X_{an}, called the *analytification* of X. As a set X_{an} equals $X(\mathbb{C})$. The topology and structure sheaf are defined as follows.

First assume that X is affine. Then X is of the form $\mathrm{Spec}(\mathbb{C}[x_1,\ldots,x_n]/I)$; we then have a natural inclusion $X(\mathbb{C}) \to \mathbb{C}^n$. We endow $X(\mathbb{C})$ with the subspace topology, where we assume that the topology on $\mathbb{C}^n$ is the Euclidean one. We let $\mathcal{H}_X = \mathcal{H}_{\mathbb{C}^n}/I\mathcal{H}_{\mathbb{C}^n}$. We find that $X_{\mathrm{an}} = (X(\mathbb{C}), \mathcal{H}_X)$ is an affine analytic space.

In general, if X is a scheme locally of finite type over $\mathbb{C}$, then X is obtained by gluing affine open subsets. The analytification X_{an} of X is then obtained by gluing the analytifications of these affine open subsets.

There is a natural map of locally ringed spaces $\phi : (X_{\mathrm{an}}, \mathcal{H}_{X_{\mathrm{an}}}) \to (X, O_X)$: the map $X_{\mathrm{an}} \to X$ is simply the inclusion $X(\mathbb{C}) \subset X$, and $O_X \to \phi_*\mathcal{H}_{X_{\mathrm{an}}}$ sends a regular function f on $U \subset X$ to the corresponding regular (and therefore holomorphic) functions on $U(\mathbb{C}) \subset X(\mathbb{C})$. For any $x \in X_{\mathrm{an}}$ the morphism of local rings $O_{X,\phi(x)} \to \mathcal{H}_{X,x}$, and therefore a morphism on the completions $\hat{O}_{X,\phi(x)} \to \hat{\mathcal{H}}_{X,x}$.

Proposition 1.4.1. *The natural morphism $\hat{O}_{X,\phi(x)} \to \hat{\mathcal{H}}_{X,x}$ is an isomorphism.* □

Let X be a scheme locally of finite type over $\mathbb{C}$. Consider the functor

$$\Phi_X : \mathbf{AnSp} \to \mathbf{Set} : Z \mapsto \mathrm{Hom}_{\mathbb{C}}(Z, X).$$

Here $\mathrm{Hom}_{\mathbb{C}}(Z, X)$ denotes the set of homomorphisms in the category of locally ringed spaces over $\mathrm{Spec}\,\mathbb{C}$.

Theorem 1.4.2. *The functor Φ_X is representable by X_{an}: composition with the morphism $\phi : X_{\mathrm{an}} \to X$ induces an isomorphism* $\mathrm{Hom}_{\mathbb{C}}(Z, X_{\mathrm{an}}) \cong \mathrm{Hom}_{\mathbb{C}}(Z, X)$. □

By the theorem, we see that every morphism $f : X \to Y$ of schemes that are of finite type over $\mathbb{C}$ lifts to a unique morphism $f_{\mathrm{an}} : X_{\mathrm{an}} \to Y_{\mathrm{an}}$ such that the following diagram commutes.

$$\begin{array}{ccc} X_{\mathrm{an}} & \xrightarrow{f_{\mathrm{an}}} & Y_{\mathrm{an}} \\ \downarrow & & \downarrow \\ X & \xrightarrow{f} & Y \end{array}$$

Proposition 1.4.3. *Let $f : X \to Y$ be a morphism of schemes of finite type over $\mathbb{C}$, let $x \in X$ be a closed point, and let $y = f(x)$. Then the diagram*

$$\begin{array}{ccc} O_{Y,y} & \longrightarrow & O_{X,x} \\ \downarrow & & \downarrow \\ \mathcal{H}_{Y_{\mathrm{an}},y} & \longrightarrow & \mathcal{H}_{X_{\mathrm{an}},x} \end{array}$$

is cartesian, with faithfully flat vertical arrows.

Proof. Don't know yet. □

1.5 The comparison theorem

Let X be a $\mathbb{C}$-scheme locally of finite type, and let X_{an} be its analytification. We define the site X_{cet} as the category of local isomorphisms $U \to X$, and a family $\{U_i \to U\}$ of morphisms in X_{cet} is a covering if it is jointly surjective. If $Y \to X$ is an étale morphism then $Y_{\mathrm{an}} \to X_{\mathrm{an}}$ is a local isomorphism, and we obtain a continuous map $X_{\mathrm{cet}} \to X_{\mathrm{ét}}$.

Theorem 1.5.1 (Comparison theorem)**.** *Let $f : X \to S$ be a morphism of finite type of schemes locally of finite type over $\mathbb{C}$, so that we have a commutative diagram of continuous maps of sites*

$$\begin{array}{ccc} X_{\mathrm{cet}} & \xrightarrow{\epsilon} & X_{\text{ét}} \\ \downarrow{\scriptstyle f_{\mathrm{cet}}} & & \downarrow \\ S_{\mathrm{cet}} & \xrightarrow{\epsilon} & S_{\text{ét}}. \end{array}$$

If F is a sheaf of sets (resp. sheaf of ind-finite groups (?), resp. torsion sheaf) on $X_{\text{ét}}$, and one of the following conditions holds:

- *F is constructible;*
- *f is proper,*

then the natural maps

$$\epsilon^*(R^q f_{\text{ét}*}F) \to (R^q f_{\text{cet}*}\epsilon^* F)$$

are bijective for $q = 0$ (resp. $q = 0, 1$, resp. $q \geq 0$).