Mastermath Elliptic Curves, Homework 4
Due: 13th October 2015, 10:15

Students are expected to (try to) solve all problems below. The ones marked as "Homework" are to be handed in and count towards your grade according to the rules on the web page.

All problems and their solutions are part of the course, and could play a role in the exam. More importantly, they help you digest the material of the previous lecture and help you prepare for the next lecture.

Problem 22 (Homework)**.** Show that being isogenous is an equivalence relation on the set of complex tori, and that there are uncountably many isogeny classes of complex tori.

Problem 23 (Homework)**.** The *multiplicator ring* of a lattice Λ is defined as

$$\mathcal{O}(\Lambda) = \{\alpha \in \mathbb{C} \mid \alpha\Lambda \subset \Lambda\}.$$

(a) Show that $\mathcal{O}(\Lambda)$ is a subring of $\mathbb{C}$ isomorphic to the endomorphism ring $\mathrm{End}(\mathbb{C}/\Lambda)$ of the torus $\mathbb{C}/\Lambda$, that is, the ring of isogenies from $\mathbb{C}/\Lambda$ to itself.

(b) Show that we have $\mathcal{O}(\Lambda) = \mathbb{Z}$ unless Λ is homothetic to a lattice of the form $\mathbb{Z} + \mathbb{Z}\,\lambda$, with $\lambda \in \mathbb{C} \setminus \mathbb{R}$ a zero of an irreducible quadratic polynomial $aX^2 + bX + c \in \mathbb{Z}[X]$, and that in this exceptional case we have $\mathcal{O}(\Lambda) = \mathbb{Z}\big[\frac{D+\sqrt{D}}{2}\big]$ with $D = b^2 - 4ac < 0$.

[In the exceptional case, we say that $\mathbb{C}/\Lambda$ has *complex multiplication* by $\mathcal{O}(\Lambda)$.]

Problem 24. Show that the subrings of $\mathbb{C}$ that are lattices correspond bijectively to the set of negative integers D satisfying $D \equiv 0, 1 \pmod 4$ under the association $D \mapsto \mathcal{O}(D) = \mathbb{Z}\big[\frac{D+\sqrt{D}}{2}\big]$. Show that there exists a ring homomorphism $\mathcal{O}(D_1) \to \mathcal{O}(D_2)$ if and only if D_1/D_2 is a square in $\mathbb{Z}$.

Problem 25 (Homework)**.** Compute the structure of the group $\mathrm{Hom}(\mathbb{C}/\Lambda_1, \mathbb{C}/\Lambda_2)$ for each of the following choices of Λ_1 and Λ_2:

(a) $\Lambda_1 = \Lambda_2 = \mathbb{Z} + \mathbb{Z}\,i$;

(b) $\Lambda_1 = \mathbb{Z} + \mathbb{Z}\,i$ and $\Lambda_2 = \mathbb{Z} + \mathbb{Z}\,2i$;

(c) $\Lambda_1 = \mathbb{Z} + \mathbb{Z}\,i$ and $\Lambda_2 = \mathbb{Z} + \mathbb{Z}\sqrt{-2}$.

Problem 26 (Homework)**.** Define the j-invariant of a lattice Λ by

$$j(\Lambda) := \frac{g_2(\Lambda)^3}{g_2(\Lambda)^3 - 27g_3(\Lambda)^2}.$$

Prove that, if Λ and Λ' are homothetic lattices, then $j(\Lambda) = j(\Lambda')$.

Problem 27. For $\tau \in \mathbb{C}$ with $\Im(\tau) > 0$, define $j(\tau) := j(\mathbb{Z} + \mathbb{Z}\,\tau)$. Prove that j is a holomorphic function of τ.